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Toroidally Symmetric Polynomial Multipole Solutions of the Vector Laplace Equation

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A coherent method is given for generating to arbitrary order, the toroidally symmetric, polynomial multipole solutions of the vector Laplace (Grad–Shafranov operator) equation. In a source-free region, the toroidal component of a toroidally symmetric magnetic vector potential may be conveniently expanded in terms of these multipoles which at large aspect ratio reduce to the simple cylindrical form $(X+iZ)^m$. The set of multipoles considered in previous work is shown to be incomplete and additional ones are derived which partially resolve this difficulty. The expansion technique is criticized, and several practical examples are given.

1. INTRODUCTION

We consider the toroidally symmetric multipole solutions of the toroidal component of the vector Laplace equation $\nabla \times \nabla \times \mathbf{A} = 0$. When expressed in terms of the poloidal magnetic flux this equation is $\Delta^* \Psi = 0$, where Δ^* is the Grad–Shafranov operator. Of all the possible forms of multipole solutions of this equation, one set is particularly conveient for the expansion of arbitrary solutions in a finite, source free region. In cylindrical coordinates the multipoles of this set are finite positive order polynomials. In contrast to other expansion functions, they are easily visualized, physically interpreted, and rapidly calculated.

As shown by Okada *et al.* [1] and others, the relation between each multipole and plasma shape and position is simple. The Z symmetric dipole term yields the uniform vertical equilibrium magnetic field necessary to counter the radial hoop force of a plasma and determines its radial position. The Z antisymmetric dipole term yields a radial field which determines the vertical position of the plasma. The quadrapole terms determine the plasma ellipticity, the hexapole terms the triangularity, and so on. This feature is convenient in the initial phase of machine

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design as one may optimize the plasma shape and size without going into the details of the coil structure. In fact, given the known effect that each multipole has on plasma shape and position, coil systems are often designed to produce given polynomial multipoles [2] yielding simple scaling and shaping rules for design purposes, and improved control over plasma shape and position during machine operation.

The PEST 2-D ideal MHD, free boundary equilibrium code can optionally use either thin ring coils or a multipole expansion to define the applied poloidal field. Not much computational savings is had with multipoles over thin ring coils as the applied flux need only be given on the boundary of the computation region. Further, it may be calculated at the beginning and stored. The bulk of the computation time is taken up by solving the internal problem with these given boundary conditions. In instances where applied field is varied, for example when a particular magnetic axis is desired, some minor savings may be obtained. The two methods of defining the applied field are otherwise equivalent in the computation as plasma parameters vary. However, coils which lie too near the solution region cannot be modeled with multipoles. In this case a combination of both methods sometimes can be used. The main reason for the implementation of a multipole model in the code is their previously mentioned utility in machine design.

Another example of the utility of multipoles is the theoretical analysis of equilibria of Greene *et al.* [5] based on aspect ratio orderings, where a simple expression for the applied fields is needed.

It is often desirable to know the contribution of eddy currents to magnetics measurements. The precise locations and values of image currents may be unknown yet their effect can often be expressed in terms of these multipoles. These eddy current fields may be defined by giving the transfer function (step function) response of each multipole component of the field when excited by a given coil group. These transfer functions can be determined by making appropriate measurements. The field which results from some arbitrary combination of image currents is then quickly calculated without resort to a detailed and often inaccurate model of the structure in which they flow. The Princeton Beta Experiment (PBX) surface analysis code SURFAS sucessfully implements a transfer function eddy current model via multipoles. This code runs between shots and cannot afford the time or space necessary to use a more complex model.

Multipoles are especially useful in the diagnosis of plasma equilibria. Multipole moments of equilibrium quantities such as pressure and toroidal current density may be related to poloidal field measurements via the divergence theorem. Zakharov and Shafranov [3] originated this method considering only Z symmetric multipoles. Shkarofsky [4] generalized it to include antisymmetric multipoles.

Although a number of authors [1–7] have either exhibited or given derivations of partial sets of these multipoles, the multipoles considered by them are not complete. An arbitrary solution of the vector Laplace equation, subject to the natural boundary conditions of the physical problem, cannot be fully expressed as a series in the multipoles of their set. In this work we derive another set of multipoles which partially resolve this difficulty and allow an arbitrary field to be more accurately approximated with a multipole expansion.

The plan of this paper is as follows. In Section 2, We give a new, more general derivation of the multipoles considered by previous authors, together with a standard norm. The derivation applies to multipoles of arbitrary order and symmetry about the midplane. It connects the multipoles to well known solutions of the vector Laplace equation, and suggests the existence of an independent set of multipoles.

In Section 3, We develop this additional set of multipoles using the derivation of Section 2. We give a nominal argument for their necessity in the expansion of an arbitrary field by exhibiting a related expansion for the Green's function of the problem.

In Section 4, We study a commonly used expansion in terms of the multipoles with regard to its validity in a practical device. We find that an improvement in error results when the augmented set of multipoles is used.

Finally, we give the algebraic form of the first few of the multipoles and their conjugate functions in a table at the end of this article.

2. Derivation of the Multipoles

Here we use three coordinate systems, spherical (ρ, θ, ϕ), cylindrical (R, ϕ, Z), where $R = \rho \sin \theta$ and $Z = \rho \cos \theta$ and cartesian (X, Z), where $X = R - R_0$. R_0 is a given arbitrary expansion point, typically the canonical major radius of a toroidal device (Fig. 1). The principle determining characteristic of these axisymmetric, ϕ independent multipoles is that at large aspect ratio, $(X^2 + Z^2)/R_0^2 \ll 1$, they reduce to the form of simple cartesian multipoles

$$\frac{(X + iZ)^m}{R_0^{m-2}}, \tag{1}$$

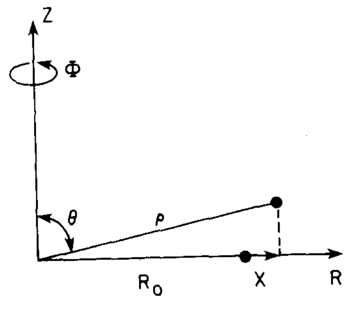

FIG. 1. Coordinate systems used are spherical (ρ, θ, ϕ), cylindrical (R, ϕ, Z), and cartesian (X, Z), where $X = R - R_0$.

$m>0$ is the cartesian multipole number. The denominator is chosen to get convenient dimensions.

The poloidal magnetic field may be defined in a source free region by

$$\mathbf{B}_{\text{pol}} = \frac{\nabla\phi \times \nabla\Psi}{2\pi} = \frac{\nabla\Phi}{2\pi}. \tag{2}$$

Here $\Psi = -2\pi R A_\phi$ is the poloidal magnetic flux and Φ is the magnetic potential conjugate to Ψ.

In cylindrical coordinates and assuming toroidal symmetry $(\partial/\partial\phi = 0)$, $\nabla \times \mathbf{B}_{\text{pol}} = 0$ and $\nabla \cdot \mathbf{B}_{\text{pol}} = 0$ yield respectively the homogeneous Grad–Shafranov operator equation for Ψ

$$\Delta^*\Psi = \left(R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}\right)\Psi = 0 \tag{3}$$

and Laplace's equation for Φ

$$\nabla^2\Phi = \left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}\right)\Phi = 0. \tag{4}$$

At large aspect ratio both of the above equations reduce to the canonical cartesian Laplace equation in X and Z whose solutions are (1). This suggests that appropriate linear combinations of the solutions of (3) and (4) can be separately found which also reduce to the form of (1). In spherical coordinates (3) and (4) are

$$\rho^2\frac{\partial^2\Psi}{\partial\rho^2} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) = 0 \tag{5}$$

and

$$\frac{\partial}{\partial\rho}\rho^2\frac{\partial\Phi}{\partial\rho} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0. \tag{6}$$

The equation for Ψ is then satisfied for $n \geqslant 2$ by

$$F_n(\rho, \theta) = C_n\rho^n \sin\theta P^1_{n-1}(\cos\theta) \tag{7}$$

or

$$H_n(\rho, \theta) = D_n\rho^{1-n} \sin\theta P^1_{n-1}(\cos\theta). \tag{8}$$

For $n \geqslant 2$ the conjugate Φ are respectively

$$G_n(\rho, \theta) = nC_n\rho^{n-1}P_{n-1}(\cos\theta) \tag{9}$$

or

$$I_n(\rho, \theta) = -(n-1)\, D_n \rho^{-n} P_{n-1}(\cos\theta). \tag{10}$$

The P_n^m are the associated Legendre functions. C_n and D_n are arbitrary constants.

The poloidal flux is subject to the physically natural homogeneous boundary conditions

$$\lim_{\theta \to 0} \Psi(\rho, \theta) = \lim_{\theta \to \pi} \Psi(\rho, \theta) = 0.$$

We therefore do not include the $Q_n(\cos\theta)$ solutions of (5) which diverge logarithmically for $\theta \to 0$ and $\theta \to \pi$. We must further have

$$\lim_{\rho \to 0} \Psi(\rho, \theta) = \lim_{\rho \to \infty} \Psi(\rho \cdot \theta) = 0.$$

This implies that expansions in both the F_n and the H_n are needed in appropriate regions of space to everywhere express Ψ. We neglect momentarily the H_n solutions which depend on negative powers of ρ and discuss their inclusion in Section 3. Previous studies of the multipoles did not consider the latter functions.

The P_n^m are generated by the Rodrigues' formula [8]

$$P_n(\chi) = \frac{(-1)^n}{2^n n!} \frac{d^n}{d\chi^n} (1-\chi^2)^n \tag{11}$$

and by

$$P_n^m(\chi) = (-1)^m (1 \times \chi^2)^{m/2} \frac{d^m}{d\chi^m} P_n(\chi). \tag{12}$$

The F_n, when expressed in cylindrical (R, Z) coordinates, are finite positive integer order polynomials. We normalize the F_n so that the coefficient of the high order term in R is equal to unity. That is, for odd n,

$$C_n = \frac{-2^{n-1}(-1)^{(n+1)/2}((n+1)/2)!((n-3)/2)!}{(n+1)!} \tag{13}$$

and for even n

$$C_n = \frac{2^{n-1}(-1)^{n/2}(n/2)!((n-2)/2)!}{n!}. \tag{14}$$

This defines the F_n and G_n for all $n \geqslant 2$. To obtain the desired large aspect ratio expansions we need to define appropriate $n < 2$ extensions of formulae (7) through (10). These may be made up out of the primitive solutions of equations (5) (i.e., Constant, ρ, $\cos\theta$) and (6) (i.e., Constant, $Q_0(\cos\theta)$, $1/\rho$). We select those which give simple results.

The first few F_n in cylindrical coordinates are then

$$\begin{aligned} &F_0 = 1,\\ &F_1 = 0,\\ &F_2 = R^2,\\ &F_3 = R^2 Z,\\ &F_4 = R^2(R^2 - 4Z^2),\\ &F_5 = (R^2 Z(3R^2 - 4Z^2))/3,\\ &F_6 = R^2(R^4 - 12R^2Z^2 + 8Z^4). \end{aligned} \tag{15}$$

The F_n and G_n can also be obtained by substitution of a series of the appropriate form into Eqs. (3) and (4), respectively [4]. In general series form the F_n are for $n > 2$,

$$F_n(R, Z) = \sum_{m=0}^{n-2} A_m^n R^{n-m} Z^m. \tag{16}$$

Here $A_0^n = 1$ and $A_1^n = 0$ if n is even, $A_0^n = 0$ and $A_1^n = 1$ if n is odd. For $2 \leqslant m \leqslant n-2$

$$A_m^n = -\frac{(n+2-m)(n-m)}{m(m-1)} A_{m-2}^n. \tag{17}$$

The conjugate G_n functions which are needed [3, 4] for calculation of moments of the toroidal current density and obey

$$R\partial G_n/\partial Z = -\partial F_n/\partial R$$

and

$$R\partial G_n/\partial R = \partial F_n/\partial Z, \tag{18}$$

are similarily given by

$$\begin{aligned} &G_0 = 0,\\ &G_1 = -1, \end{aligned}$$

and for $n \geqslant 2$,

$$G_n = \frac{Z}{R^2} \sum_{m=0}^{n-2} \left(\frac{(n-m)}{(m+1)} A_m^n R^{n-m} Z^m \right) - \left(\frac{R^{n-1}}{n-1} \right)_{n\,\mathrm{odd}}. \tag{19}$$

The last term appears only if n is odd.

The F_n do not have the desired large aspect ratio form. The linear combinations of the F_n which yield the cylindrical multipoles upon expansion about R_0 are given in a generalized form, which we will later apply to the G_n, H_n, and I_n, for even n by

$$\Psi_n^+ = \frac{2\pi}{m2^m} \sum_{l=0}^{m} (-1)^{m-l} R_0^{2-2l} \binom{m}{l} F_{2l}(R, Z) \tag{20}$$

and for odd n by

$$\Psi_n^+ = \frac{2\pi}{m2^m} \sum_{l=1}^{m} (-1)^{m-1} R_0^{1-2l} \binom{m}{l} 2l F_{2l+1}(R, Z) - \frac{2\pi \Sigma_n}{m2^m}. \tag{21}$$

Here $\binom{m}{l}$ are the binomial coefficients and m is the cartesian multipole number associated with n. For even and odd n respectively

$$\begin{aligned} m &= n/2, \\ m &= (n-1)/2. \end{aligned} \tag{22}$$

Σ_n in Eq. (21) is a constant term which cancels the preceeding sum when $R = R_0$ and $Z = 0$. It vanishes when Eq. (21) is applied to the F_n. The large aspect ratio expansion of Eqs. (20) and (21) yields for even n

$$\Psi_n^+ + i\Psi_{n+1}^+ = \frac{2\pi(X + iZ)^m}{m R_0^{m-2}}. \tag{23}$$

Our choice of normalization for the Ψ_n^+ is now justified firstly by the fact that $\Psi = B_v \Psi_2^+$ yields a uniform vertical field of strength B_v, and second in that the asymptotic form of Ψ_n^+ from Eq. (23) yields $|\mathbf{B}_{\text{real}}(\sqrt{X^2 + Z^2} = R_0)| = 1$.

Equations (20) and (21) generalize the low order multipoles given in the references [1–7] to all orders, and both even and odd symmetry about $Z = 0$, and provide a straightforward method for their derivation.

The algebraic form of the first few of the Ψ_n multipoles are given in the table at the end of this article. They were generated using the REDUCE [9] algebraic manipulations program on a DEC10 computer. The reduce input file, which allows one to generate the algebraic form of these multipoles to large, computer dependent order, and Fortran subroutines for numerically evaluating them, are obtainable from the authors.

3. A Further Set of Multipoles

The symmetric Greens function for the Grad–Shafranov operator which vanishes along the Z axis and at $\rho = \infty$, obeys

$$\Delta_1^* G(R_1, Z_1, R_2, Z_2) = 2\pi R_1 \delta(R_1 - R_2)\, \delta(Z_1 - Z_2). \tag{24}$$

In terms of elliptic integrals it is

$$G(R_1, Z_1, R_2, Z_2) = -\frac{(R_1 R_2)^{1/2}}{\kappa}((2-\kappa^2) K(\kappa) - 2E(\kappa)),$$

where

$$\kappa^2 = \frac{4R_1 R_2}{((R_1+R_2)^2 + (Z_1 - Z_2)^2)}. \tag{25}$$

This Greens function can be shown to have the spherical coordinate expansion

$$G(\rho_1, \theta_1, \rho_2, \theta_2) = -\pi\rho_1 \sin\theta_1 \rho_2 \sin\theta_2 \sum_{l=1}^{\infty} \frac{\rho_<^l}{\rho_>^{l+1}} \frac{P_l^1(\cos\theta_1) P_l^1(\cos\theta_2)}{l(l+1)}. \tag{26}$$

Here $\rho_>$ is the larger of and $\rho_<$ the smaller of ρ_1 and ρ_2.

This expansion does not involve the Q_n due to our imposed boundary conditions, but does depend on both positive and negative powers of ρ. The field due to any distribution of toroidal current can be found by integrating this expanded Greens function times the distribution. The expansion functions form a complete set for the expression of any such field in an appropriately restricted domain. Obviously we need the H_n functions as well as the F_n functions to form a complete set in this domain. If we let $D_n = R_0^{2n-1} C_n$ then

$$\begin{aligned} H_0 &= (R^2 + Z^2)^{1/2}/R_0, \\ H_1 &= 0, \\ H_n &= \frac{R_0^{2n-1} F_n(R, Z)}{(R^2+Z^2)^{n-1/2}}. \end{aligned} \tag{27}$$

The linear combinations of Eqs. (20) and (21) applied to the H_n, with $\Sigma_n = 0$, then yield an independent set of Ψ_n^- polynomial multipoles which give upon expansion for even n

$$\Psi_n^- + i\Psi_{n+1}^- = \frac{2\pi(iZ - X)^m}{mR_0^{m-2}} \tag{28}$$

and asymptotically redundant contributions to the set of cylindrical multipoles.

The Φ_n^+ and Φ_n^- conjugate functions to the Ψ_n^+ and Ψ_n^- are also easily found by applying the linear combinations of Eqs. (20) and (21) to the G_n and I_n functions of Eqs. (9) and (10) given the additional definitions

$$\begin{aligned} I_0 &= \frac{1}{2R_0} \log\left(\frac{1-\cos\theta}{1+\cos\theta}\right), \\ I_1 &= -R_0 \end{aligned} \tag{29}$$

and that

$$\Sigma_n = (-1)^m R_0, \tag{30}$$

when applying Eq. (21) to the G_n while

$$\Sigma_n = \frac{-(-2)^m (m)!\, R_0}{n!!}, \tag{31}$$

when applying Eq. (21) to the I_n functions.

The linear combinations of Eq. (20) and (21) when applied to the F_n and G_n are equivalent to those of Shkarofsky [4], whose odd n F_n and G_n functions differ from those given here by a factor of $2m$. Equation (21) does not however depend on the F_1, H_1, G_1, and I_1 functions which are only given here for completeness. The Ψ_n^- and Φ_n^- functions thus defined are an equivalent set to the Ψ_n^+ and Φ_n^+ in so far as the diagnostic technique of Zakharov and Shafranov [3] is concerned. Their use ought to provide additional information about the current distribution at a small increase in labor with possibly critical consequences if the information being sought is the value of the poloidal beta. This generalized approach to magnetics diagnostics will not be developed here.

4. A Multipole Expansion Technique

The validity of a multipole expansion technique for the expression of an applied poloidal field is studied in what follows by some examples taken from the Princeton Divertor Experiment (PDX). This technique has been used at Princeton Plasma Physics Laboratory to help design PBX, TFTR, TFCX, and ISX-B. Here we are not attempting to solve for the poloidal field in a given region but rather finding if such an expansion can accurately represent an actual field as generated by the PDX coil systems.

As has been stated, the field due to any distribution of toroidal current can be reduced to a multipole-like expansion in some source free region of interest by integrating over all space the form of the distribution times the expanded Greens function of Eq. (26) and truncating the resultant series. For most practical coil systems the series derived in this manner is inconvenient as the coils do not lie outside of some region of interest bounded by spheres. The Green's function derived series for a general coil system is in fact several series each of whose domain of applicability lies in the region between sucessive spherical coil radii. The Ψ^+ multipoles are complete only in a source free region inside a sphere of finite radius, since they are linear combinations of $\rho^n P_{n-1}^1$. They can be applied in a strict sense only to those devices whose coil systems have this internal hole. Including the Ψ^- functions yields a complete set of functions for a source free spherical annulus. With n resticted to an integer so as to obtain a finite series expression for the multipoles, the inclusion of the Q_n functions would not result in a complete set for domains restricted in θ. Even if n were allowed to be complex, the domain in which the P_n and Q_n functions were complete would be of inconvenient shape.

The multipole expansion technique seeks to replace the Green's function series with one which is asymptotically valid and close to correct at finite aspect ratio. This expansion assumes a priori that the coefficients of F_n and H_n are constant over the domain of interest. These coefficients can be thought of as an average of the coefficients of the Green's function expansion. The global validity of this assumption of constancy is not obvious for coils which have intermediate radii. However, since the large aspect ratio limit is accurate, the analytic continuation of this limit satisfies the operator equation, and for some coils the series is valid we may expect to obtain a satisfactory degree of accuracy in a finite region of arbitrary shape for many coil systems.

The necessity for this assumption of constancy arises because we are trying to expand a field in a toroidal region with functions which are originally derived in spherical coordinates. The trick we have played is to give them at least a partial toroidal character by forming appropriate linear combinations. Other expansion methods exist which do not have this intrinsic problem, notably those in terms of toroidal functions. While the toroidal functions are the more appropriate choice for some problems they are not as convenient as the polynomial multipole expansion. Further, their convergence rate may be no better in cases which trouble the latter.

General error criteria for the multipole expansion process are difficult to give as they depend on the between coil spacing and the resultant fringing fields. However an expansion in only the Ψ_n^+ positive power of ρ multipoles cannot in general be expected to well represent those coils which lie on spherical radii less than the radius of the region of interest. We therefore study the error introduced by this technique in an actual device.

The PDX device has several independently powered coil systems, among them are the equilibrium field (EF) system which produces an almost uniform vertical field which counters the radial hoop force of the toroidal plasma current, the divertor (DF) system which is dominantly octopole and produces poloidal field nulls near the plasma surface, and the ohmic (OH) system which drives the toroidal plasma current and in the vicinity of the plasma gives a weak leakage field of mixed character. A further contribution to the applied poloidal field is that of the eddy currents induced in the vacuum vessel and coil cans. For the present comparison these were not measured but instead inferred from a slab metallic elements model image current code.

In each case we numerically generate the exact poloidal flux in a subregion of the PDX device due to a given coil system using an accurate numerical approximation to the Greens function of Eq. (25). We then fit to obtain the coefficients B_n^+, B_n^- of a multipole expansion of the form

$$\Psi(R, Z) = \sum_{n=0,2,4,\ldots}^{N} B_n^+ \Psi_n^+(R, Z, R_0) + B_n^- \Psi_n^-(R, Z, R_0). \tag{32}$$

We pick as our expansion point $R_0 = 1.4$ meters, the nominal center of PDX and choose our subregion to have an aspect ratio $A \approx 3.0$, and consider only the multipoles with even symmetry about the midplane. We compare the results of first assuming all the B_n^- to be zero and truncating the series at $N = 16$ (hexadecapole) to those obtained by truncating the series at $N = 8$ (octapole), but including both terms.

While our expansion functions are linearly independent, they are not orthogonal when their products are integrated over the sub-region except at large aspect ratio, where the Ψ_n^+ and Ψ_n^- become degenerate sets of orthogonal functions. At finite aspect ratio a simple least squares fit of (32) while giving a small overall error which decreases with the truncation number of the expansion, yields coefficients which fluctuate with truncation number as pairs of high order multipoles subtract to give low order contributions confusing the intepretation of which multipoles are dominant. Regularization of the simple least squares fit by finding that expansion point R_0 which minimizes some spectral measure of the expansion does not substantially improve this fluctuation.

A fitting method which avoids this problem and gives intuitively agreeable results as to which multipoles are dominant in the field proceeds as follows:

I. A constant equal to the exact flux at R_0 is subtracted from the function being fitted.

II. The appropriate amount of dipole multipole (either Ψ_2^+ or Ψ_2^- can be used) which brings the field null to R_0 is subtracted.

III. The resultant function is individually fitted in turn to each of the multipoles of order $n = 4$ (quadrapole) or greater in our expansion set. The particular multipole which gives the best fit in the least squares sense and the smallest mean square error is singled out and subtracted to give a residual function. This best fit multipole is then deleted from the expansion set.

IV. This residual function is then analyzed as in step III using the remainder of the functions in the expansion set. Each multiple is individually fit to the residual function. That single multipole which now best fits the residual function is subtracted from it and deleted from the expansion set. Step IV is then repeated until all of the multipoles in the expansion set are exhausted.

This fitting procedure is equivalent to first doing Gram–Schmidt orthogonalization of the expansion functions in a particular order and then doing a simultaneous least squares fit. In practice it extracts the multipoles in an order which agrees with intuition as to their relative dominance in the field pattern. The final error obtained is equivalent to that gotten with a simultaneous least squares fit. At large aspect ratio, the multipoles of positive and negative power of ρ dependence become degenerate. The fitting method still works but the selection of multipoles of one set over the other is arbitrary.

Usually the minimum error is almost achieved with the extraction only a few (two or three) multipoles of order $n = 4$ or higher, the extraction of the remaining multipoles in the expansion set then reduces the error only slightly. With the exception of the DF system, for reasons discussed later, the flux due to all of the PDX coil systems may be fit to maximum errors of around one percent and mean square errors of a few tenths of a percent. However, for all of the coil systems the inclusion of the Ψ_n^- multipoles reduces both errors by a factor of two, even when the same total number of expansion functions is used, as might have been expected from the completeness property of the combined set. In what follows we quote only the results due to the combined fit, and use as a convenient measure of the field strength due to each multipole, the half aspect ratio quantity $(2/A)^{m-1}B_n$. We normalize by the dominant multipole field strength and give the power of ρ dependence as a sign in parentheses.

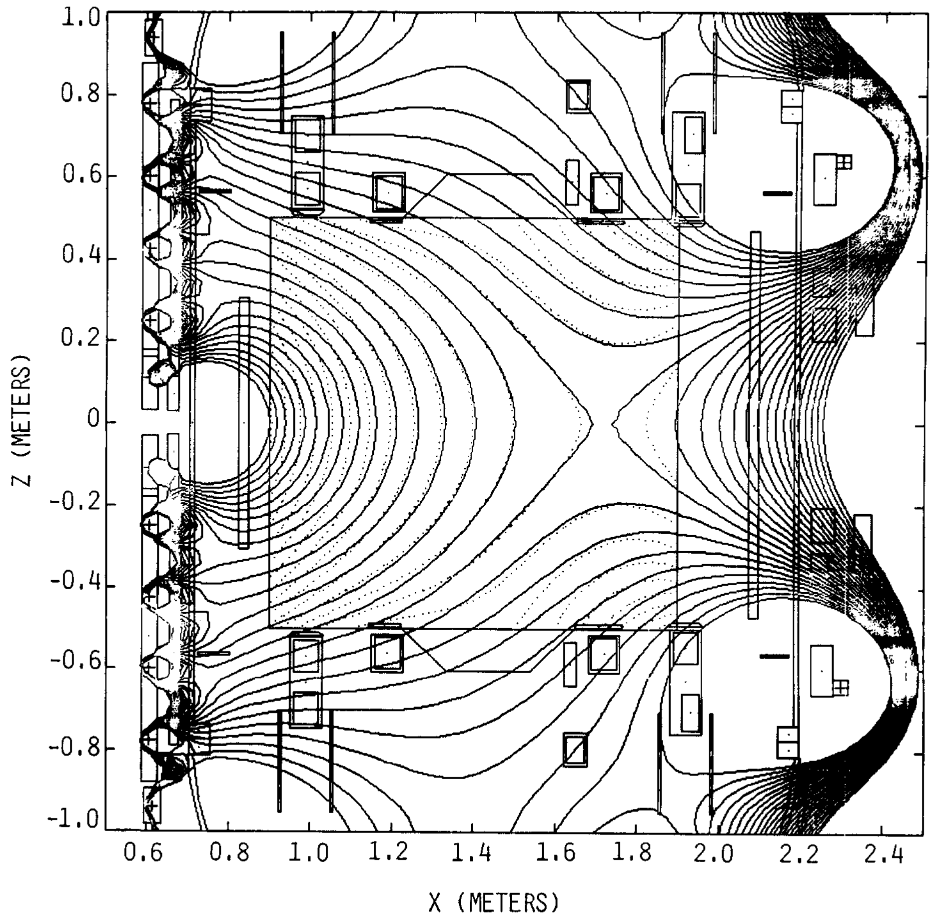

FIG. 2. The ohmic coil system (OH) leakage field. The box in the center is the subregion in which the fit to the poloidal flux is made. Solid lines are contours of the exact flux while dotted lines are contours of the multipoles expansion fit at the same values.

The EF field is well represented by the expansion with a root mean square error of two tenths of one percent and is dominantly dipole (1.0) in character with small contributions of (+) quadrapole (−0.16) and (+) hexapole (0.013).

Figure 2 shows the leakage field due to the ohmic heating solenoid. The subregion considered is the box in the center of the plot. The solid lines are contours at equal flux increments of the exact flux, while the dotted lines are contours of the multipoles fit at the same values. The relatively large displacement of the multipoles expansion contours from their exact counterparts os due to the weak field gradient as the rms error is small (3.5*e*-04) and the field is well represented. The ohmic leakage field is dominantly (+) quadrapole (1.0) with (+) dipole contributions of −0.39 and (−) quadrapole of 0.39.

Figure 3 shows an equivalent plot for the PDX divertor coil system (DF) which

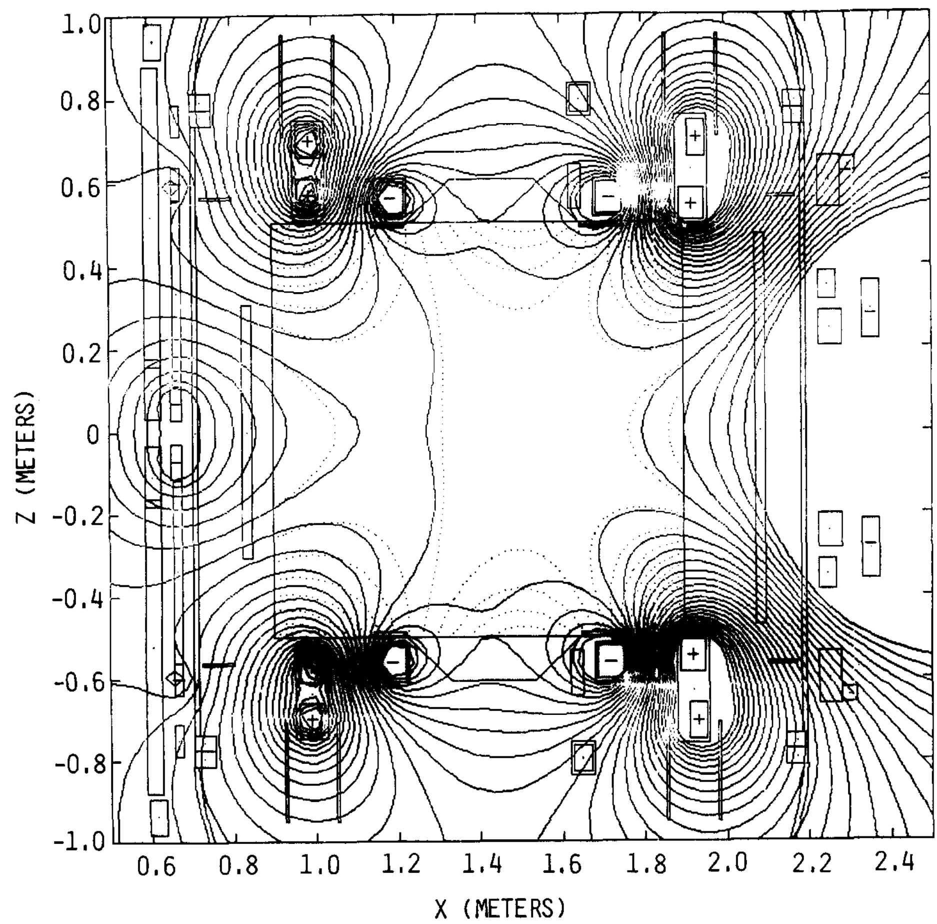

FIG. 3. The divertor coil system (DF) field when compared with the best fit multipoles expansion shows the inadaquacy of the latter when coils are too near the expansion region.

is purposely included here as an example of poor performance of the multipole expansion method. The rms error is large (0.2), and the field is mostly (+) octopole (1.0), with some (−) octopole (−0.18) and (−) hexapole (0.16). This field is poorly represented because the subregion lies too near the coils which produce this field and the multipole expansion can converge, if at all, only slowly, i.e., as $(\rho_{<}/\rho_{>})^n$. The error could be reduced by shrinking the subregion but the plasmas of interest soon would not fit in. In reaction design this problem may not occur as poloidal field coils must usually lie outside the toroidal field coils. However designs which depend on very strong shaping of the plasma as, for example, the bean shaped plasma of some recent devices, could not be studied solely with a multipole expansion method.

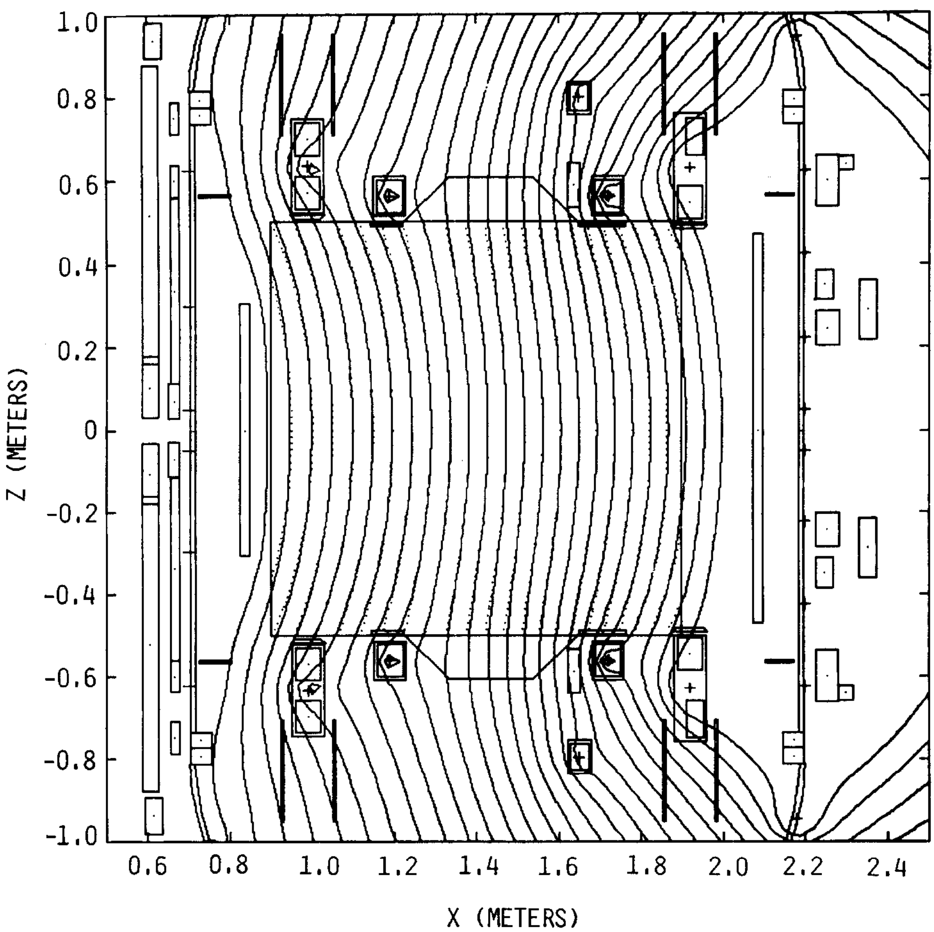

FIG. 4. The magnetic field produced by image currents at $t = 50$ ms in a PDX shot as calculated with a slab metallic elements model eddy current code (solid lines) and the multipoles expansion fit (dotted lines).

TABLE I

Multipoles

$\Psi_0^+ = (2\pi)\, R_0^2$, even nullapole
$\Phi_0^+ = 0$

$\Psi_1^+ = 0$, odd nullapole
$\Phi_1^+ = -(2\pi)\, R_0$

$\Psi_2^+ = (2\pi)(R^2 - R_0^2)/2$, even dipole
$\Phi_2^+ = -(2\pi)\, Z$

$\Psi_3^+ = (2\pi)(R^2 E)/R_0$, odd dipole
$\Phi_3^+ = (2\pi)(R^2 - 2Z^2 - R_0^2)/(2R_0)$

$\Psi_4^+ = (\pi)(R^4 - 4R^2Z^2 - 2R^2R_0^2 + R_0^4)/(4R_0^2)$, even quadrapole
$\Phi_4^+ = (\pi)\, Z(-3R^2 + 2Z^2 + 3R_0^2)/(3R_0^2)$

$\Psi_5^+ = (\pi)\, R^2 Z(3R^2 - 4Z^2 - 3R_0^2)/(3R_0^3)$, odd quadrapole
$\Phi_5^+ = (\pi)(3R^4 - 24R^2Z^2 - 6R^2R_0^2 + 8Z^4$
$+ 12Z^2R_0^2 + 3R_0^4)/(12R_0^3)$

$\Psi_6^+ = (2\pi|3)(R^6 - 12R^4Z^2 - 3R^4R_0^2 + 8R^2Z^4$
$+ 12R^2Z^2R_0^2 + 3R^2R_0^4 - R_0^6)/8R_0^4)$, even hexapole

$\Phi_6^+ = (2\pi/3)\, Z(-15R^4 + 40R^2Z^2 + 30R^2R_0^2 - 8Z^4$
$- 20Z^2R_0^2 - 15R_0^4)/(20R_0^4)$

$\Psi_7^+ = (2\pi/3)\, R^2 Z(15R^4 - 60R^2Z^2 - 30R^2R_0^2 + 24Z^4$
$+ 40Z^2R_0^2 + 15R_0^4)/(20R_0^5)$, odd hexapole

$\Phi_7^+ = (2\pi/3)(5R^6 - 90R^4Z^2 - 15R^4R_0^2 + 120R^2Z^4$
$+ 120R^2Z^2R_0^2 + 15R^2R_0^4 - 16Z^6 - 40Z^4R_0^2$
$- 30Z^2R_0^4 - 5R_0^6)/(40R_0^5)$

$\Psi_8^+ = (\pi/2)(5R^8 - 120R^6Z^2 - 20R^6R_0^2 + 240R^4Z^4$
$+ 240R^4Z^2R_0^2 + 30R^4R_0^4 - 64R^2Z^6 - 160R^2Z^4R_0^2$
$- 120R^2Z^2R_0^4 - 20R^2R_0^6 + 5R_0^8)/(80R_0^6)$, even octopole
$\Phi_8^+ = (\pi/2)\, Z(-35R^6 + 210R^4Z^2 + 105R^4R_0^2 - 168R^2Z^4$
$- 280R^2R_0^2 - 105R^2R_0^4 + 16Z^6 + 56Z^4R_0^2$
$+ 70Z^2R_0^4 + 35R_0^6)/(70R_0^6)$

$\Psi_9^+ = (\pi/2)\, R^2 Z(35R^6 - 280R^4Z^2 - 105R^4R_0^2 + 336R^2Z^4$
$+ 420R^2Z^2R_0^2 + 105R^2R_0^4 - 64Z^6 - 168Z^4R_0^2$
$- 140Z^2R_0^4 - 35R_0^6)/(70R_0^7)$, odd octopole
$\Phi_9^+ = (\pi/2)(35R^8 - 1120R^6Z^2 - 140R^6R_0^2 + 3360R^4Z^4$
$+ 2520R^4Z^2R_0^2 + 210R^4R_0^4 - 1792R^2Z^6 - 3360R^2Z^4R_0^2$
$- 1680R^2Z^2R_0^4 - 140R^2R_0^6 + 128Z^8 + 448Z^6R_0^2$
$+ 560Z^4R_0^4 + 280Z^2R_0^6 + 35R_0^8)/(560R_0^7)$

$\Psi_{10}^+ = (2\pi/5)(7R^{10} - 280R^8Z^2 - 35R^8R_0^2 + 1120R^6Z^4$
$+ 840R^6Z^2R_0^2 + 70R^6R_0^4 - 896R^4Z^6 - 1680R^4Z^4R_0^2$
$- 840R^4Z^2R_0^4 - 70R^4R_0^6 + 128R^2Z^8 + 448R^2Z^6R_0^2$
$+ 560R^2Z^4R_0^4 + 280R^2Z^2R_0^6 + 35R^2R_0^8 - 7R_0^{10})/(224R_0^8)$, even decapole
$\Phi_{10}^+ = (2\pi/5)\, Z(-315R^8 + 3360R^6Z^2 + 1260R^6R_0^2 - 6048R^4Z^4$
$- 7560R^4Z^2R_0^2 - 1890R^4R_0^4 + 2304R^2Z^6 + 6048R^2Z^4R_0^2$
$+ 5040R^2Z^2R_0^4 + 1260R^2R_0^6 - 128Z^8 - 576Z^6R_0^2$
$- 1008Z^4R_0^4 - 840Z^2R_0^6 - 315R_0^8))/(1008R_0^8)$

Table continued

TABLE I—*Continued*

$\Psi_{11}^{+} = (2\pi/5)\, R^2Z(315R^8 - 4200R^6Z^2 - 1260R^6R_0^2 + 10080R^4Z^4$
$+ 10080R^4Z^2R_0^2 + 1890R^4R_0^4 - 5760R^2Z^6 - 12096R^2Z^4R_0^2$
$- 7560R^2Z^2R_0^4 - 1260R^2R_0^6 + 640Z^8 + 2304Z^6R_0^2$
$+ 3024Z^4R_0^4 + 1680Z^2R_0^6 + 315R_0^8)/(1008R_0^9)$, odd decapole

$\Phi_{11}^{+} = (2\pi/5)(63R^{10} - 3150R^8Z^2 - 315R^8R_0^2 + 16800R^6Z^4$
$+ 10080R^6Z^2R_0^2 + 630R^6R_0^4 - 20160R^4Z^6 - 30240R^4Z^4R_0^2$
$- 11340R^4Z^2R_0^4 - 630R^4R_0^6 + 5760R^2Z^8 + 16128R^2Z^6R_0^2$
$+ 15120R^2Z^4R_0^4 + 5040R^2R_0^6 + 315R^2R_0^8 - 256Z^{10}$
$- 1152Z^8R_0^2 - 2016Z^6R_0^4 - 1680Z^4R_0^6 - 630Z^2R_0^8$
$- 63R_0^{10})/(2016R_0^9)$

$\Psi_0^- = (2\pi)\, \rho R_0$, even nullapole

$\Phi_0^- = (2\pi)\, I_0(\theta)\, R_0^2$

$\Psi_1^- = 0$, odd nullapole

$\Phi_1^- = -(2\pi)\, R_0$

$\Psi_2^- = (2\pi)\, R_0(-\rho^4 + R^2R_0^2)/(2\rho^3)$, even dipole

$\Phi_2^- = (2\pi)(-I_0(\theta)\, R_0^2\rho^3 + ZR_0^3)/(2\rho^3)$

$\Psi_3^- = (2\pi)(R^2ZR_0^4)/(\rho^5)$, odd dipole

$\Phi_3^- = (2\pi)\, R_0(\rho^5 - R^2R_0^3 + 2Z^2R_0^3)/(3\rho^5)$

$\Psi_4^- = (\pi)\, R_0(\rho^8 - 2R^6R_0^2 - 4R^4Z^2R_0^2 + R^4R_0^4$
$- 2R^2Z^4R_0^2 - 4R^2Z^2R_0^4)/4\rho^7)$, even quadrapole

$\Phi_4^- = (\pi)(I_0(\theta)\, R_0^2\rho^7 + ZR_0^3(-2\rho^4$
$+ 3R^2R_0^2 - 2Z^2R_0^2)/(4\rho^7)$

$\Psi_5^- = (\pi)\, R^2ZR_0^4(-3\rho^4 + 3R^2R_0^2 - 4Z^2R_0^2)/(3\rho^9)$, odd quadrapole

$\Phi_5^- = (\pi)\, R_0(-2\rho^8 + 5R^6R_0^3 - 3R^4R_0^5 - 15R^2Z^4R_0^3$
$+ 24R^2Z^2R_0^5 - 10Z^6R_0^3 - 8Z^4R_0^5)/(15\rho^9)$

$\Psi_6^- = (2\pi/3)\, R_0(-\rho^{12} + 3R^{10}R_0^2 + 12R^8Z^2R_0^2 - 3R^8R_0^4$
$+ 18R^6Z^4R_0^2 + 6R^6Z^2R_0^4 + R^6R_0^6 + 12R^4Z^6R_0^2$
$+ 21R^4Z^4R_0^4 - 12R^4Z^2R_0^6 + 3R^2Z^8R_0^2 + 12R^2Z^6R_0^4$
$+ 8R^2Z^4R_0^6)/(8\rho^{11})$, even hexapole

$\Phi_6^- = (2\pi/3)(-3I_0(\theta)\, R_0^2\rho^{11} + ZR_0^3(9\rho^8$
$- 27R^6R_0^2 - 36R^4Z^2R_0^2 + 15R^4R_0^4 + 9R^2Z^4R_0^2$
$- 40R^2Z^2R_0^4 + 18Z^6R_0^2 + 8Z^4R_0^4))/(24\rho^{11})$

Note. The normalization factor of this paper is given for convenience as the first term in every formula.

Figure 4 illustrates the multipole expansion as applied to the eddy current fields arising in PDX from currents flowing in the vacuum cans which surround the divertor coils and the vacuum vessel early ($t = 50$ ms) in a PDX shot. The rms error is small ($5.0e-03$) and the field is (+) dipole (1.0) plus (+) quadrapole (-0.41) and ($-$) quadrapole (-0.07).

5. Conclusions

In conclusion, a new and coherent method has been given for deriving the Ψ_n^+ multipole solutions of the homogeneous Grad–Shafranov operator equation to arbitrary order for both even and odd symmetry. This method connects the multipoles to well-known solutions of the vector Laplace's equation and this connection has resulted in the derivation of a new set of Ψ_n^- multipoles. Together with the previously known set these provide a more accurate basis for the expansion of arbitrary solutions of the homogeneous Grad–Shafranov operator equation subject to natural boundary conditions. This new set may be useful in magnetic diagnosis of the equilibrium state of the plasma. The new functions are of definite use in the approximation of applied poloidal fields as an expansion in terms of these multipoles, where they allow a more accurate representation to be made.

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